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LETTER TO THE EDITOR

The Ising spin glass in a magnetic field

David Elderfield

Solid State Theory Group, The Blackett Laboratory, Imperial College, London SW7 2BZ, UK

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Abstract. The profile of the field-cooled magnetisation is considered in the vicinity of the spin-glass transition for the Parisi solution. In a finite field the flat profile of Parisi and Toulouse is modified and one observes a weak cusp. Analytic expansions are obtained both for the high-field (low-temperature) and low-field (high-temperature) regimes.

The Parisi solution (Parisi 1979, 1980) of the Sherrington–Kirkpatrick (SK) Ising spin glass (Sherrington and Kirkpatrick 1975), recently interpreted in terms of the metastable states of Thouless *et al* (1977) and Bray and Moore (1981) by de Dominicis and Young (1982), Dasgupta and Sompolinsky (1983) and Parisi (1983), is now widely accepted. We describe here the field-cooled magnetisation profile in the vicinity of the Almeida–Thouless line (Almeida and Thouless 1978), comparing our predictions with those of the Parisi and Toulouse (1980) hypothesis, as appropriate. In a finite field the magnetisation is not simply related to the local susceptibilities discussed by numerous authors (Parisi 1979, Sompolinsky 1981), so our results represent an important addition to the spin-glass literature. The analysis will be performed within the *equivalent* Sompolinsky (1981) framework (de Dominicis *et al* 1982) which offers certain technical advantages.

The SK model is defined for Ising spins $S_i = \pm 1$ by the Hamiltonian

$$H = -\tfrac{1}{2} \sum_{i,j=1}^{N} J_{ij} S_i S_j - h \sum_{i=1}^{N} S_i \tag{1}$$

where the J_{ij} are quenched independent gaussian random exchanges of infinite range with zero mean and variance J^2/N. Using the dynamical approach of Sompolinsky (1981) or the replica analysis of de Dominicis *et al* (1982) the free energy F per spin in the thermodynamic limit may be formulated in terms of the extremal problem

$$\beta F = \mathrm{ext}[\beta F(\{q(x), \Delta(x)\})]_{\Delta(1)=0} \tag{2}$$

where the free energy functional $F(\{q(x), \Delta(x)\})$ is defined as follows

$$-\beta F(\{q(x), \Delta(x)\}) = \frac{\beta^2}{4}\left((1-q(1))^2 + 2\int_0^1 \mathrm{d}x\ \Delta'(x) q(x)\right)$$
$$+\overline{\log(\cosh(\beta H)) + \frac{\beta^2}{2}\int_0^1 \mathrm{d}x\ \Delta'(x)[M]_x}, \tag{3}$$

in terms of an effective field H

$$H \equiv h + z\sqrt{q}(0) + \int_0^1 \mathrm{d}x \; (z(x)\sqrt{q'}(x) - \beta\Delta'(x)[M]_x), \tag{4}$$

and effective magnetisation M

$$M = \tanh(\beta H). \tag{5}$$

Here a bar above denotes averaging over the gaussian random variables $z(x)$, $x \in (0, 1)$ and z for which

$$\overline{z(x)} = 0 = \bar{z}, \qquad \overline{z(x)z(x')} = \delta(x - x'), \qquad \overline{z^2} = 1, \tag{6}$$

whilst $[\ldots]_x$ defines a restricted average over the variables $z(y)$, $y > x$. We have chosen units such that $J = 1$. Variation of (3) with respect to $q'(x)$, $\Delta'(x)$, $x \in (0, 1)$ leads directly to the Sompolinsky equations

$$\partial(-\beta F)/\partial q'(x) = \tfrac{1}{2}\beta^2(q(1) - 1 - \Delta(x) + \overline{\partial M/\partial(\beta h(x))}) = 0, \tag{7}$$

$$\partial(-\beta F)/\partial\Delta'(x) = \tfrac{1}{2}\beta^2(q(x) - \overline{[M]_x^2}) = 0, \tag{8}$$

where we define $h(x) \equiv \sqrt{q'}(x)z(x)$, whilst variation with respect to $q(0)$ simply reproduces (7) for $x \to 0$.

In the present framework the field-cooled (FC) or thermodynamic magnetisation m is given by the familiar relation

$$m \equiv -\frac{\partial F}{\partial h} = \begin{cases} \bar{M} \\ \sum_{i=1}^{N} \langle S_i \rangle, \end{cases} \tag{9}$$

where the angle brackets denote a full Gibbs statistical average. On the other hand the identification of the zero-field cooled (ZFC) magnetisation requires a more detailed understanding of the interplay between the Parisi–Sompolinsky solution and the metastable states of Thouless *et al* (1977), and is at present unknown. We shall, however, identify several useful susceptibilities. First the global FC or thermodynamic susceptibility given by the relation

$$\chi(\mathrm{FC}) \equiv \frac{\mathrm{d}m}{\mathrm{d}h} = \begin{cases} \overline{\mathrm{d}M/\mathrm{d}h} \\ \dfrac{\beta}{N} \sum_{i,j=1}^{N} \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle, \end{cases} \tag{10}$$

should be compared with the local FC susceptibility

$$\chi^l(\mathrm{FC}) = \frac{\partial m}{\partial h} = \begin{cases} \overline{\partial M/\partial h} = \beta(1 - q(1) + \Delta(0) \qquad \text{see (7)} \\ \dfrac{\beta}{N} \sum_{i=1}^{N} (1 - \langle S_i \rangle^2). \end{cases} \tag{11}$$

Here the angle brackets denote a full Gibbs statistical average and we use the total derivative in (10) to emphasie that $\chi(\mathrm{FC})$ includes contributions from the explicit response of $q(x)$, $\Delta(x)$ to the field h, so that typically $\chi(\mathrm{FC})$, $\chi^l(\mathrm{FC})$ are distinct†. To

† The limit $h \to 0$ is exceptional for the gauge symmetry $S_i \to -S_i$, $J_{ij} \to -J_{ij}$ all j any i ensures $\chi(\mathrm{FC}) = \chi^l(\mathrm{FC})$ for $h = 0$.

obtain the ZFC susceptibilities is more difficult. At present only the local ZFC susceptibility has been identified and is given by

$$\chi^{\mathrm{l}}(\mathrm{ZFC})=\begin{cases}\overline{\partial M/\partial h}=\beta(1-q(1)) & \text{(see (7))}\\ \dfrac{\beta}{N}\displaystyle\sum_{i=1}^{N}(1-\langle S_i\rangle_{\mathrm{R}}^2)\end{cases} \tag{12}$$

where $\langle\ldots\rangle_{\mathrm{R}}$ denotes a restricted Gibbs average appropriate to the statistical mechanics of a single metastable state.

To describe the vicinity of the Almeida–Thouless critical line $T_{\mathrm{AT}}(h)$ which marks the onset of spin-glass behaviour or anomalous response in the (h, T) plane it is useful to follow Parisi (1980), de Dominicis *et al* (1982) and Sommers (1983) and reformulate the free energy functional ((3) *et seq*) in the form

$$-\beta F(\{q(x),\Delta(x)\})=\frac{\beta^2}{4}\left((1-q(1))^2+2\int_0^1 \mathrm{d}x\ \Delta'(x)q(x)\right)+\overline{\phi(y=h+\sqrt{q}(0)z,0)}, \tag{13}$$

where the function $\phi(y, x)$, defined as follows

$$\phi(y,x)=\overline{\log[\cosh(\beta H(x))]+\frac{\beta^2}{2}\int_x^1 \mathrm{d}r\ \Delta'(r)[M(x)]_r},$$

$$H(x)\equiv y+\int_x^1 \mathrm{d}r\ (z(r)\sqrt{q'}(r)-\beta\Delta'(r)[M(x)]_r), \qquad M(x)\equiv\tanh(\beta H(x)), \tag{14}$$

satisfies the nonlinear integral equation

$$\phi(y,x)-\log(\cosh(\beta y))=\tfrac{1}{2}\beta^2\int_x^1 \mathrm{d}r\ [q'(r)(\partial^2\phi(y,r)/\partial y^2)-\Delta'(r)(\partial\phi(y,r)/\partial y)^2]. \tag{15}$$

It is important to realise that in a sense the relations (8), (9) are incomplete. Analysing the structure of F, it is evident from the alternative representation ((13) *et seq*) that the system is invariant under arbitrary reparametrisations $x\rightarrow f(x)$, f monotonic, so that predictions are limited to the endpoints of the natural interval chosen as (0, 1). Here $x=0$ characterises the FC state and $x=1$ the statistical mechanics of the ZFC or 'metastable' state. The Parisi solution in fact corresponds to a 'gauge' choice $\Delta'(x)=-xq'(x)$ which, interpreted by Hertz (1983) in terms of an effective cluster dynamics, fixes x uniquely (Sommers 1983), provided we use the symmetry $x\rightarrow f(x)$ to eliminate the plateaux $(q'(x)=0)$ which are therefore apparently artifacts of the replica approach. Here we shall restrict attention to the experimentally accessible limits $x=0, 1$ for which such subtleties are largely irrelevant.

To solve (2) or (7), (8), below and in the vicinity of the AT line $T_{\mathrm{AT}}(h)$, or throughout the high temperature $(T>T_{\mathrm{AT}}(h))$ paramagnetic phase we develop an effective free energy functional, valid if the irreversibility estimated by $|\Delta(x)|$, $|q(1)-q(x)|$ is small. Constructing $\phi(y, x)$ and whence F, iteratively we find that the free energy functional $F(\{q(x), \Delta(x)\})$ is of the form

$$-\beta F=\frac{\beta^2}{4}\left((1-q(1))^2+2\int_0^1 \mathrm{d}x\ \Delta'(x)q(x)\right)+\bar{\psi} \tag{16}$$

where the functional ψ is given explicitly by the relation

$$
\begin{aligned}
\psi &\equiv \phi(h+\sqrt{q}(0)z,0)\\
&= \log\{\cosh[\beta(h+\sqrt{q}(0)z)]\}+\tfrac{1}{2}\beta^2[(1-t^2)g(0)+\Delta(0)]\\
&\quad -\frac{\beta^4}{4}\left[(1-t^2)(1-3t^2)g^2(0)-2(1-t^2)^2\int_0^1 \mathrm{d}x\ \Delta'(x)g(x)\right]\\
&\quad +\beta^6\left[(1/3!)(1-t^2)(2-15t^2+15t^4)g^3(0)+(1-t^2)^2(1-3t^2)\right.\\
&\quad \left.\times\int_0^1 \mathrm{d}x\ (\Delta(x)-\Delta(0))g(x)\Delta'(x)-(1-t^2)^2(1-5t^2)g(0)\right]\\
&\quad \times\int_0^1 \mathrm{d}x\, g(x)\Delta'(x)-t^2(1-t^2)^2\int_0^1 \mathrm{d}x\, g^2(x)\Delta'(x)\Big)+\mathrm{O}(g^4,\Delta^4)
\end{aligned}
\tag{17}
$$

in agreement with Sommers (1983). Here $g(x)\equiv q(1)-q(x)-\Delta(x)$, $t=\tanh[\beta(h+\sqrt{q}(0)z]$ and the bar above refers to the remaining Gaussian average over $z(\bar{z}=0,\ \overline{z^2}=1)$. In the high-temperature paramagnetic phase $(T>T_{\mathrm{AT}}(h))$, both $q'(x)$, $\Delta'(x)$ are zero and (16) reduces to the SK or replica symmetric free energy $F_{\mathrm{SK}}(q)$

$$
F_{\mathrm{SK}}(q)=(\beta^2/4)(1-q)^2+\overline{\log\{\cosh[\beta(h+\sqrt{q}z)]\}}. \tag{18}
$$

Variation of F (13), (14) or (16), (17) shows directly that the Sompolinsky equations (7), (8) may be written in the form

$$
q(x)=\overline{\left(\frac{\partial\psi}{\partial(\beta h)}\right)^2}+\beta^2\int_0^x \mathrm{d}r\, A(\{q(r),\Delta(r)\},r)q'(r) \tag{19}
$$

$$
1-q(1)+\Delta(x)=\overline{\frac{\partial^2\psi}{\partial(\beta h)^2}}+\beta^2\int_0^x \mathrm{d}r\, A(\{q(r),\Delta(r)\},r)\Delta'(r) \tag{20}
$$

where ψ is defined by (17) and the functional A is given explicitly by the relations

$$
\begin{aligned}
A(x) &\equiv \overline{[\partial M/\partial(\beta h(x))]_x^2}\\
&= \overline{(1-t^2)^2}+2\beta^2(\overline{(1-t^2)^2(1-3t^2)}(\Delta(x)-\Delta(0))-\overline{(1-t^2)^2(1-5t^2)}g(0)\\
&\quad -\overline{2t^2(1-t^2)^2}g(x))+\mathrm{O}(\Delta^2,g^2).
\end{aligned}
\tag{21}
$$

Differentiating the *exact* relations (19), (20) with respect to x we see directly that

$$
\beta^2 A(x)=1 \qquad \text{or} \qquad q'(x)=\Delta'(x)=0. \tag{22}
$$

This constraint on the extrema of F is very useful for, in the spin glass phase $(T<T_{\mathrm{AT}}(h))$, the 'gauge' symmetry $\{x\to f(x),\ f \text{ monotonic}\}$ allows us to assume that $q'(x)$, $\Delta'(x)$ are non-zero throughout the chosen interval $(0,1)$. For example (21)

implies directly that *in the spin-glass phase*

$$1 = \beta^2 A(0) = \overline{(\partial^2\psi/\partial h^2)^2}$$

$$= \beta^2\overline{(1-t^2)^2} - 2\beta^4\overline{(1-t^2)^2(1-3t^2)}g(0) + \beta^6\Big(\overline{(1-t^2)^2(5-36t^2+39t^4)}g^2(0)$$

$$-4\overline{(1-t^2)^3(1-5t^2)}\int_0^1 \mathrm{d}x\ \Delta'(x)g(x)\Big) + \mathrm{O}(g^2, \Delta^3), \tag{23}$$

where for $x \to 0$ we can identify $A(x)$ in terms of ψ, which is known to $\mathrm{O}(g^3, \Delta^3)$ from (17). In addition, differentiating the constraint (22) with respect to x, we obtain from (21) the relation

$$0 = (\partial/\partial x)(A(x)) = 2\beta^2[\overline{(1-t^2)^2(1-3t^2)}\Delta'(x) - \overline{2t^2(1-t^2)^2}g'(x)] + \mathrm{O}(g^2, \Delta^2), \tag{24}$$

which we will use to eliminate $\Delta'(x)$, $g(x)$ $(x \neq 0)$ from the problem up to corrections of order τ^3 where $\tau \equiv 1 - T/T_{\mathrm{AT}}(h)$ is the appropriate reduced temperature.

To determine the magnetisation (9), we observe that (16) implies directly that

$$m = \overline{\partial\psi/\partial h} = \bar{t} - \beta^2[\overline{t(1-t^2)g(0)}] + \beta^4\Big[\overline{t(1-t^2)(1-3t^2)}g^2(0)$$

$$-\overline{2t(1-t^2)^2}\int_0^1 \mathrm{d}x\ \Delta'(x)g(x)\Big] + \mathrm{O}(g^3, \Delta^3), \tag{25}$$

which through the function 't', depends explicitly on the parameter $q(0)$, given selfconsistently by the extremum equation ((19), $x = 0$). To the required order we have

$$q(0) = \overline{\left(\frac{\partial\psi}{\partial h}\right)^2} = \overline{t^2} - 2\beta^2[\overline{t^2(1-t^2)}g(0)] + \beta^4\Big[\overline{t^2(1-t^2)(5-7t^2)}g^2(0)$$

$$-\overline{4t^2(1-t^2)^2}\int_0^1 \mathrm{d}x\ \Delta'(x)g(x)\Big] + \mathrm{O}(g^3, \Delta^3). \tag{26}$$

Using (23), (25), (26), we now see directly that the Almeida–Thouless temperature $T_{\mathrm{AT}}(h)$ which marks the onset of spin-glass behaviour or anomalous response $(q'(x), -\Delta'(x) > 0)$ is determined by the familiar relation

$$1 = \beta^2\int_{-\infty}^{+\infty}\frac{\mathrm{d}z}{\sqrt{2\pi}}\exp(-z^2/2)\,\mathrm{sech}^4[\beta(h+\sqrt{q_{\mathrm{SK}}}z)]. \tag{27}$$

where q_{SK} is given by the SK expression ((26), $q(0) = q_{\mathrm{SK}}$, $\Delta'(x) = g'(x) = 0$ and $g(1) = 0$). For $T > T_{\mathrm{AT}}(h)$, (25) trivialises and we recover the SK expression

$$m = m_{\mathrm{SK}} \equiv \int_{-\infty}^{+\infty}\frac{\mathrm{d}z}{\sqrt{2\pi}}\exp(-z^2/2)\tanh[\beta(h+\sqrt{q_{\mathrm{SK}}}z)], \tag{28}$$

recently discussed by Bouchiat (1983). In the spin-glass phase the situation is more complicated. Qualitatively we observe near T_{AT}

$$g(0), \Delta(0) \sim \tau > 0 \qquad q(0) \sim \begin{Bmatrix} 1 \\ h^{2/3} \end{Bmatrix} \qquad \text{for} \qquad \begin{Bmatrix} h \gg 1, & T_{\mathrm{AT}} \sim \exp(-h^2) \\ h \ll 1, & 1 - T_{\mathrm{AT}} \sim h^{2/3} \end{Bmatrix}, \tag{29}$$

so that the structure of the solution depends critically on the magnitude of h $(\tau \ll 1)$. To make analytical progress we shall restrict our attention to the vicinity of the Almeida-Thouless line $T_{\mathrm{AT}}(h)$, for high fields $h \gg 1$ where $q_{\mathrm{SK}} \sim 1$, so that the AT line is given by the relation (see below)

$$T_{\mathrm{AT}} = \tfrac{4}{3}(2\pi)^{-1/2}\exp(-\tfrac{1}{2}h^2)[1 + T_{\mathrm{AT}}^2(h^2-1)(\pi^2/24-1)] + \mathrm{O}[T_{\mathrm{AT}}(T_{\mathrm{AT}}h)^4] \tag{30}$$

and for low fields $h \ll 1$ where $q_{\mathrm{SK}} \sim h^{2/3}$, leading to the equation (see below)

$$(1-T_{\mathrm{AT}})^3 = \tfrac{3}{4}h^2[1-(\tfrac{3}{4}h^2)^{1/3}] + \mathrm{O}(h^{7/3}). \tag{31}$$

To analyse the low-field magnetisation it is useful to rewrite (23), (25) and (26) in the form

$$\begin{aligned} m = \bar{\theta} = h[&\beta\theta^{(1)} + (\beta^3/3!)\theta^{(3)}(h^2+3q(0)) + (\beta^5/5!)\theta^{(5)}(h^4+10h^2q(0)+15q^2(0)) \\ &+ (\beta^7/7!)\theta^{(7)}(105q^3(0)+\ldots] \end{aligned} \tag{32}$$

$$\begin{aligned} 1 = \beta^2 A(0) = \overline{(\partial\theta/\partial h)^2} = \beta^2(&(\theta^{(1)})^2 + \beta^2\theta^{(1)}\theta^{(3)}(h^2+q(0)) + \beta^4((\theta^{(3)}/4)^2 + \theta^{(1)}\theta^{(5)}/12) \\ &\times[3(h^2+q(0))^2 - 2h^4] \\ &+ \beta^6(\theta^{(3)}\theta^{(5)}/4! + 2\theta^{(1)}\theta^{(7)}/6!)[15(h^2+q(0))^3+\ldots] \end{aligned} \tag{33}$$

$$\begin{aligned} q(0) = \overline{\theta^2} = \beta^2(\theta^{(1)})^2(h^2+q(0)) &+ \tfrac{1}{3}\beta^4\theta^{(1)}\theta^{(3)}[3(h^2+q(0))^2 - 2h^4] \\ &+ \beta^6[(\theta^{(3)}/3!)^2 + 2\theta^{(1)}\theta^{(5)}/5!][15(h^2+q(0))^3 - 15(h^2+2q(0))h^4] \\ &+ 2\beta^8(\theta^{(3)}\theta^{(5)}/3!\,5! + \theta^{(1)}\theta^{(7)}/7!)(105q^4(0)+\ldots) \end{aligned} \tag{34}$$

where the functions $\theta[\beta(h+\sqrt{q}(0)z)]$, $\theta^{(2k+1)}$ are defined as follows

$$\theta = \partial\psi/\partial(\beta h) = \sum_{k=0} \frac{1}{(2k+1)!}\theta^{(2k+1)}[\beta(h+\sqrt{q}(0)z]^{2k+1}. \tag{35}$$

To obtain the parameters $\theta^{(2k+1)}$ recursively, we express the function θ in the form

$$\begin{aligned} \theta = \partial\psi/\partial(\beta h) = t + \tfrac{1}{2}\beta^2 t^{(2)}g(0) + \beta^4\Big(\tfrac{1}{8}t^{(4)}g^2(0) + (\tfrac{1}{6}t^{(2)} - \tfrac{1}{24}t^4)\int_0^1 \mathrm{d}x\, \Delta'(x)g(x)\Big) \\ + \mathrm{O}(g^3, \Delta^3) \end{aligned} \tag{36}$$

where $t^{(k)} = (\partial^k/\partial(\beta h)^k)\tanh[\beta(h+\sqrt{q}(0)z)]$. Most generally therefore

$$\begin{aligned} \theta^{(k)} = \frac{\partial^k}{\partial y^k}\theta(y)\Big|_{y=0} &= a^{(k)} + \tfrac{1}{2}\beta^2 a^{(k+2)}g(0) \\ &+ \beta^4\Big(\tfrac{1}{8}a^{(k+4)}g^2(0) + (\tfrac{1}{6}t^{(k+2)} - \tfrac{1}{24}t^{(k+4)})\int_0^1 \mathrm{d}x\, \Delta'(x)g(x)\Big) + \mathrm{O}(g^3, \Delta^3) \end{aligned} \tag{37}$$

where now $a^{(k)} = (\partial^k/\partial y^k)\tanh(y)|_{y=0}$ or explicitly $a^{(1)} = 1$, $a^{(3)} = -2$, $a^{(5)} = 16$ and $a^{(2k)} = 0$ for all k. Now for small fields $h \ll 1$ near the AT line, we expect $q(0) \sim q_{\mathrm{SK}}(T_{\mathrm{AT}}) \sim h^{2/3}$ so (32), (33), (34) imply that

$$\begin{aligned} m/h = 1 - \tfrac{1}{4}\beta^6(\theta^{(3)})^2(q(0)+h^2)^2[&1 + (\theta^{(5)}/\theta^{(3)})\beta^2 q(0) - \tfrac{1}{8}(\theta^{(3)})^2\beta^6 q^2(0)] \\ &- \tfrac{1}{3}\beta^3\theta^{(3)}h^2(1 + \tfrac{1}{2}(\theta^{(5)}/\theta^{(3)})\beta^2 q(0)) + \mathrm{O}(h^{10/3}). \end{aligned} \tag{38}$$

$$\begin{aligned} h^2 = \tfrac{1}{3}\beta^6(\theta^{(3)})^2(q(0)+h^2)^3(1 &+ (\theta^{(5)}/\theta^{(3)})\beta^2 q(0) + (\tfrac{7}{10}(\theta^{(5)}/\theta^{(3)})^2 \\ &+ \tfrac{1}{4}(\theta^{(7)}/\theta^{(3)}))(\beta^2 q(0))^2) + \mathrm{O}(h^4) \end{aligned} \tag{39}$$

where the parameters $\theta^{(k)}$, depend through $g(0)$ on the reduced temperature $\tau = 1 - T/T_{\mathrm{AT}}(h)$. Now solving (33) for $g(0)$, using (36), (37), (38) and (39) we find that

$$g(0) = \tau\left(1 - \tfrac{5}{3}(\tfrac{3}{4}h^2)^{1/3}\right) + \mathrm{O}(\tau^3, \tau h^{4/3}) \tag{40}$$

vanishes continuously as the Almeida–Thouless temperature $T_{\mathrm{AT}}(h)$ (31) is approached from the spin-glass phase. If we introduce (40) into (38), (36) and (37) we are then finally led to the following expression for the magnetisation ($\tau > 0$)

$$\begin{aligned} m/h = 1 - (\tfrac{3}{4}h^2)^{2/3}\{&[1 + \tfrac{32}{9}(\tfrac{3}{4}h^2)^{1/3} - \tfrac{7189}{90}(\tfrac{3}{4}h^2)^{2/3}] - \tfrac{2}{3}\tau[1 + 14(\tfrac{3}{4}h^2)^{1/3}] + \tfrac{37}{9}\tau^2\} \\ &+ \mathrm{O}(\tau^3, h^{10/3}) \end{aligned} \tag{41}$$

which is correct to the *second non-trivial* order in the natural expansion parameters τ, $h^{2/3}$. At leading order (41) reduces to the celebrated Parisi (1980) expression. Our result bounds for the first time the corrections to the Parisi and Toulouse (1980) hypothesis, which states that throughout the spin-glass phase

$$m(T, h) = m(T_{\mathrm{AT}}(h), h). \tag{42}$$

We may therefore estimate the departure from that profile to be of the form

$$[m(T, h) - m(T_{\mathrm{AT}}(h), h)]/m(T, h) \sim \tfrac{8}{3}(\tfrac{3}{4}h^2)^{2/3}\tau > 0. \tag{43}$$

Bouchiat (1983) has argued on the basis of the SK magnetisation (28) and the Parisi and Toulouse hypothesis that, in small fields, the magnetisation m attains its maximum value exactly on the Almeida–Thouless line. Beyond leading order we see from (43) that this is *incorrect* and we observe instead a weak cusp in m, for in the high-temperature phase it is straightforward to show from (28) that for $\tau \ll 1 - T_{\mathrm{AT}}$

$$m/h = 1 - (\tfrac{3}{4}h^2)^{2/3}\{[1 + \tfrac{32}{9}(\tfrac{3}{4}h^2)^{1/3}] - \tfrac{107}{6}\tau\} + \mathrm{O}(\tau^2, h^{5/3}). \tag{44}$$

To analyse the high-field (low-temperature) magnetisation it is helpful to rephrase (23), (25) and (26) in the form

$$m = \bar{t} + \tfrac{1}{2}(\beta^2 g(0))\bar{t}^{(2)} + (\beta^2 g(0))^2\bar{t}^{(4)} + \tfrac{1}{6}\beta^4 \int_0^1 \mathrm{d}x\ \Delta'(x) g(x)(\bar{t}^{(2)} - \tfrac{1}{4}\bar{t}^{(4)}) + \ldots \tag{45}$$

$$\begin{aligned} 1 = \beta^2 A(0) = \beta^2\Big[&(\tfrac{2}{3}\bar{t}^{(1)} - \tfrac{1}{6}\bar{t}^{(3)}) - 2(\beta^2 g(0))^2(\tfrac{4}{15}\bar{t}^{(1)} - \tfrac{1}{6}\bar{t}^{(3)} + \tfrac{1}{40}\bar{t}^{(5)}) \\ &+ (\beta^2 g(0))^2(\tfrac{16}{21}\bar{t}^{(1)} + \ldots) + \beta^4 \int_0^1 \mathrm{d}x\ \Delta'(x) g(x)(\tfrac{16}{105}\bar{t}^{(1)} + \ldots)\Big] \end{aligned} \tag{46}$$

$$\begin{aligned} q(0) = 1 - \bar{t}^{(1)} - \tfrac{2}{3}(\beta^2 g(0))\{\bar{t}^{(1)} + \tfrac{1}{2}\bar{t}^{(3)}\} - (\beta^2 g(0))^2(\tfrac{44}{15}\bar{t}^{(1)} + \ldots) \\ - \beta^4 \int_0^1 \mathrm{d}x\ \Delta'(x) g(x)(\tfrac{8}{15}\bar{t}^{(1)} + \ldots). \end{aligned} \tag{47}$$

In this domain we expect $q(0) = q_{\mathrm{SK}} + \mathrm{O}(\tau)$, $q_{\mathrm{SK}} = 1 + \mathrm{O}(T^2)$, so we define $\delta \equiv q_{\mathrm{SK}} - q(0)$ and use the identity

$$\begin{aligned} \bar{f}(h + (a-b)^{1/2}z) &\overset{a/b<1}{=} \overline{f(h + \sqrt{a}z)} - \tfrac{1}{2}b\overline{\sqrt{a}(\partial^2 f/\partial h^2)(h + \sqrt{a}z)} \\ &+ \tfrac{1}{8}\overline{(b\sqrt{a})^2(\partial^4 f/\partial h^4)(h + \sqrt{a}z)} + \mathrm{O}((b\sqrt{a})^3) \end{aligned} \tag{48}$$

to re-express these relations in terms of $g(0)$, δ, the function $s(h, t)$ and its derivatives are defined below.

$$s = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \tanh[\beta(h+z)] = t \Big|_{q(0)=1} \\ \displaystyle\int_{-h}^{h} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} - \frac{1}{\sqrt{2\pi}} \left(\frac{\pi^2}{12}\right) T^2 h\, e^{-h^2/2} \{1 + \tfrac{7}{1440}\pi^2 (h^3 - 3) + O(T^4, (Th)^4)\} \end{cases} \tag{49}$$

$$s^{(k)} = \partial^k(h, t)/\partial h^k s = \beta^k t^{(k)}|_{q(0)=1}.$$

We find that (45), (46) and (47) may be developed in the forms

$$m = m_{SK} - \tfrac{3}{4} T_{AT} h [T_{AT}^2 (1 - 2\tau) \beta^2 g(0)(1 + \tfrac{2}{3}\beta^2 g(0)) - \delta(1 - \tfrac{3}{4}T_{AT}^2)] + \ldots \tag{50}$$

$$\tau[1 - (\tfrac{1}{12}\pi^2 - \tfrac{19}{8}) T_{AT}^2 (h^2 - 1)] = \tfrac{4}{5}\beta^2 g(0)(1 - \tfrac{9}{7}\beta^2 g(0)) + \tfrac{1}{2}\delta(h^2 - 1) + \ldots \tag{51}$$

$$\delta(1 - \tfrac{3}{4} T_{AT}^2 (h^2 - 1)) = T_{AT}^2 (1 - \tau)(\beta^2 g(0))(1 + 4(\beta^2 g(0))) + \ldots \tag{52}$$

which may be easily exploited to show that the high-field magnetisation is given by

$$m = \begin{cases} m_{SK} & T > T_{AT} \\ m_{SK} + \tfrac{45}{64} T_{AT}^3 h\tau \{T_{AT}^2 (h^2 - 2) + \tfrac{31}{9}\tau + O[\tau^2, T_{AT}^4, (T_{AT}h)^4]\} & T < T_{AT} \end{cases} \tag{53}$$

where the SK magnetisation m_{SK} may be obtained directly from (47). Explicitly

$$m_{SK} = \int_{-h}^{+h} \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} + \tfrac{1}{8} T_{AT}^3 h ((9 - \pi^2) + (\pi^2 - 9)\tau + O(\tau^2, T_{AT}^2, (T_{AT}h)^2). \tag{54}$$

The expression (53) is correct to the *second non-trivial* order in the natural expansion parameters τ, ($T_{AT}^2 \ln T_{AT}$) and T_{AT}^2. Strictly, the latter could be disregarded, for deep in the high-field domain $\ln(T_{AT}) \sim h^2 \gg 1$, however, in view of the weakness of the logarithmic singularity we shall keep such contributions. In the context of the Parisi and Toulouse hypothesis (see (42)) we see that (53), (54) predicts a difference, in this high field of the order

$$[(m(T, h) - m(T_{AT}(h), h))/m(T, h)] \sim T_{AT}^3 h\tau > 0. \tag{55}$$

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